\section*{\textbf{\large $\lambda$-BACKBONE COLORING NUMBERS OF SPLIT GRAPHS WITH TREE BACKBONES}}

A.N.M. Salman

Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung, Jalan Ganesa 10 Bandung 40132, Indonesia
msalman@math.itb.ac.id

\textbf{Abstract.} In the application area of frequency assignment graphs are used to model the topology and mutual interference between transmitters. The problem in practice is to assign a limited number of frequency channels in an economical way to the transmitter in such a way that interference is kept at an acceptable level. This has led to various different types of coloring problem in graphs. One of them is a $\lambda$-backbone coloring. Given an integer $\lambda \geq 2$, a graph $G = (V, E)$ and a spanning subgraph $H$ of $G$ (the backbone of $G$), a $\lambda$-backbone coloring of $(G, H)$ is a proper vertex coloring $V \to \{1, 2, \ldots\}$ of $G$ in which the colors assigned to adjacent vertices in $H$ differ by at least $\lambda$. The $\lambda$-backbone coloring number $BBC_\lambda(G, H)$ of $(G, H)$ is the smallest integer $\ell$ for which there exists a $\lambda$-backbone coloring $f : V \to \{1, 2, \ldots, \ell\}$. In this paper we consider the $\lambda$-backbone coloring of split graphs. A split graph is a graph whose vertex set can be partitioned into a clique (i.e. a set of mutually adjacent vertices) and an independent set (i.e. a set of mutually non adjacent vertices), with possibly edges in between. We determine sharp upper bounds for $\lambda$-backbone coloring numbers of split graphs with tree backbones.

\section{Introduction}

In [3] backbone colorings are introduced, motivated and put into a general framework of coloring problems related to frequency assignment. We refer to [3] and [2] for an overview of related research, but we repeat the relevant definitions here. For undefined terminology we refer to [1].

Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A function $f : V \to \{1, 2, 3, \ldots\}$ is a vertex coloring of $V$ if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring $f : V \to \{1, \ldots, k\}$ is called a $k$-coloring, and the chromatic number $\chi(G)$ is the smallest integer $k$ for which there exists a $k$-coloring. A set $V' \subseteq V$ is independent if $G$ does not contain edges with both end vertices in $V'$. By definition, a $k$-coloring partitions $V$ into $k$ independent sets $V_1, \ldots, V_k$.

Let $H$ be a spanning subgraph of $G$, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 2$, a vertex coloring $f$ of $G$ is a $\lambda$-backbone coloring of $(G, H)$, if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. The $\lambda$-backbone coloring number $BBC_\lambda(G, H)$ of $(G, H)$ is the smallest integer $\ell$ for which there exists a $\lambda$-backbone coloring $f : V \to \{1, \ldots, \ell\}$.

A path is a graph $P$ whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_P = \{v_1v_2, \ldots, v_{n-1}v_n\}$. A cycle is a graph $C$ whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$. A tree is a connected graph $T$ that does not contain any cycles.

A complete graph with an edge between every pair of vertices. The complete graph on $n$ vertices is denoted by $K_n$. A graph $G$ is complete $p$-partite if its vertices can be partitioned into $p$ nonempty independent sets $V_1, \ldots, V_p$ such that its edge set $E$ is formed by all edges that have one end vertex in $V_i$ and the other one in $V_j$ for some $1 \leq i < j \leq p$.

A star $S_r$ is a complete 2-partite graph with independent sets $V_1 = \{r\}$ and $V_2$ with $|V_2| = q$; the vertex $r$ is called the root and the vertices in $V_2$ are called the leaves of the star $S_r$. In our context a matching $M$ is a collection of pairwise disjoint stars that are all copies of $S_1$. We call a spanning subgraph $H$ of a graph $G$

- a tree backbone of $G$ if $H$ is a (spanning) tree;
- a star backbone of $G$ if $H$ is a collection of pairwise disjoint stars;
- a matching backbone of $G$ if $H$ is a (perfect) matching.

Obviously, $BBC_\lambda(G, H) \geq \chi(G)$ holds for any backbone $H$ of a graph $G$. In order to analyze the maximum difference between these two numbers the following values can be introduced.
\[ T_\lambda(k) = \max \{ \text{BBC}_\lambda(G, T) \mid T \text{ is a tree backbone of } G, \text{ and } \chi(G) = k \} \]
\[ S_\lambda(k) = \max \{ \text{BBC}_\lambda(G, S) \mid S \text{ is a star backbone of } G, \text{ and } \chi(G) = k \} \]
\[ M_\lambda(k) = \max \{ \text{BBC}_\lambda(G, M) \mid M \text{ is a matching backbone of } G, \text{ and } \chi(G) = k \} . \]

For the case \( \lambda = 2 \), the behavior of the first values is determined in [3] as summarized in the following result.

**Theorem 1.1.** \( T_2(k) = 2k - 1 \) for all \( k \geq 1 \).

The above theorem shows the relation between the 2-backbone coloring number and the classical chromatic number in case the backbone is a tree. The 2-backbone coloring number roughly grow like 2\( k \), where \( \chi = k \). In [4], we studied the other two cases: We first determined all values \( S_\lambda(k) \), and observed that they roughly grow like \((2 - \frac{1}{\lambda})k\). Then we determined all values \( M_\lambda(k) \) and observed that they roughly grow like \((2 - \frac{2}{\lambda + 1})k\). Their precise behavior is summarized in the two following theorems.

**Theorem 1.2.** For \( \lambda \geq 2 \) the function \( S_\lambda(k) \) takes the following values:

(a) \( S_\lambda(2) = \lambda + 1 \);  
(b) for \( 3 \leq k \leq 2\lambda - 3 \): \( S_\lambda(k) = \lfloor \frac{3k}{2} \rfloor + \lambda - 2 \);  
(c) for \( 2\lambda - 2 \leq k \leq 2\lambda - 1 \) with \( \lambda \geq 3 \): \( S_\lambda(k) = k + 2\lambda - 2 \); \( S_2(3) = 5 \);  
(d) for \( k = 2\lambda \) with \( \lambda \geq 3 \): \( S_\lambda(k) = 2k - 1 \); \( S_2(4) = 6 \);  
(e) for \( k \geq 2\lambda + 1 \): \( S_\lambda(k) = 2k - \lfloor \frac{k}{\lambda} \rfloor \).

**Theorem 1.3.** For \( \lambda \geq 2 \) the function \( M_\lambda(k) \) takes the following values:

(a) for \( 2 \leq k \leq \lambda \): \( M_\lambda(k) = \lambda + k - 1 \);  
(b) for \( \lambda + 1 \leq k \leq 2\lambda \): \( M_\lambda(k) = 2k - 2 \);  
(c) for \( k = 2\lambda + 1 \): \( M_\lambda(k) = 2k - 3 \);  
(d) for \( k = t(\lambda + 1) \) with \( t \geq 2 \): \( M_\lambda(k) = 2t\lambda - 2 \);  
(e) for \( k = t(\lambda + 1) + c \) with \( t \geq 2 \): \( 1 \leq c < \frac{k+3}{2} \): \( M_\lambda(k) = 2t\lambda + 2c - 1 \);  
(f) for \( k = t(\lambda + 1) + c \) with \( t \geq 2 \): \( \frac{k+3}{2} \leq c \leq \lambda \): \( M_\lambda(k) = 2t\lambda + 2c - 2 \).

A split graph is a graph whose vertex set can be partitioned into a clique (i.e. a set of mutually adjacent vertices) and an independent set (i.e. a set of mutually nonadjacent vertices), with possibly edges in between. The size of a largest clique and the size of a largest independent set in \( G \) are denoted by \( \omega(G) \) and \( \alpha(G) \), respectively. Split graphs were introduced by Hammer and Földes [7]; see also the book [6] by Golumbic. They form an interesting subclass of the class of perfect graphs. Hence, split graphs satisfy \( \chi(G) = \omega(G) \).

The sharp upper bounds for the \( \lambda \)-backbone coloring numbers of split graphs with star or matching backbones are determined in [5] as follows.

**Theorem 1.4.** Let \( \lambda \geq 2 \) and let \( G = (V, E) \) be a split graph with \( \chi(G) = k \geq 2 \). For every star backbone \( S = (V, E_S) \) of \( G \),

\[
\text{BBC}_\lambda(G, S) \leq \begin{cases} 
    k + \lambda & \text{if either } k = 3 \text{ and } \lambda \geq 2 \text{ or } k \geq 4 \text{ and } \lambda = 2 \\
    k + \lambda - 1 & \text{in the other cases.}
\end{cases}
\]

The bounds are tight.

**Theorem 1.5.** Let \( \lambda \geq 2 \) and let \( G = (V, E) \) be a split graph with \( \chi(G) = k \geq 2 \). For every matching backbone \( M = (V, E_M) \) of \( G \),

\[
\text{BBC}_\lambda(G, M) \leq \begin{cases} 
    \lambda + 1 & \text{if } k = 2 \\
    k + 1 & \text{if } k \geq 3 \text{ and } \lambda \leq \min\{k, \frac{k+5}{2}\} \\
    k + 2 & \text{if } k = 9 \text{ or } k \geq 11 \text{ and } \frac{k+5}{2} \leq \lambda \leq k \\
    \lfloor \frac{k}{2} \rfloor + \lambda & \text{if } k = 3, 5, 7 \text{ and } \lambda \geq k \\
    \lfloor \frac{k}{2} \rfloor + \lambda + 1 & \text{if } k = 4, 6 \text{ or } k \geq 8 \text{ and } \lambda \geq \lfloor \frac{k}{2} \rfloor + 1.
\end{cases}
\]

The bounds are tight.

In this paper we study the special case of \( \lambda \)-backbone colorings of split graphs with tree backbones. In the next section we present sharp upper bounds for the \( \lambda \)-backbone coloring numbers of split graphs with tree backbones.
2. Split graphs with tree backbones

In 2003 Broersma et al. [3] determined sharp upper bounds for the λ-backbone coloring numbers of split graphs along trees for λ = 2 as summarized in the following theorem.

**Theorem 2.1.** Let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 1$. For every tree backbone $T = (V, E_T)$ of $G$,

$$\BBC_\lambda(G, T) \leq \begin{cases} 1 & \text{if } k = 1 \\ 3 & \text{if } k = 2 \\ k + 2 & \text{if } k \geq 3. \end{cases}$$

The bound is tight.

We study λ-backbone colorings of split graphs along trees for other values of λ and generalize the result in Theorem 2.1 as follows.

**Theorem 2.2.** Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 1$. For every tree backbone $T$ of $G$,

$$\BBC_\lambda(G, T) \leq \begin{cases} 1 & \text{if } k = 1 \\ 1 + \lambda & \text{if } k = 2 \\ k + \lambda & \text{if } k \geq 3. \end{cases}$$

The bounds are tight.

**Proof of the upper bounds.** Let $G = (V, E)$ be a split graph with a spanning tree $T = (V, E_T)$. Let $C$ and $I$ be a partition of $V$ such that $C$ with $|C| = k$ is a clique of maximum size, and such that $I$ is an independent set. Since split graphs are perfect, $\chi(G) = \omega(G) = k$. The case $k = 1$ is trivial. If $k = 2$ then $G$ is bipartite, and we use colors 1 and $\lambda + 1$. For $k \geq 3$, we consider the restriction of the tree $T$ to the vertices in $C$, and we distinguish two cases.

In the first case, the restriction of $T$ to $C$ forms a star $K_{1,k-1}$. Let $v_1, \ldots, v_{k-1}$ denote the $k - 1$ leaves of this star, and let $v_k$ denote its center. For $i = 1, \ldots, k - 1$ we color $v_i$ with color $i$, and we color $v_k$ with color $k + \lambda - 1$. This yields a λ-backbone coloring for the vertices in $C$. All vertices $u \in I$ are leaves in the tree $T$. Any vertex $u \in I$ with $uv_k \notin E_T$ can be safely colored with color $k + \lambda$. It remains to consider vertices $u \in I$ with $uv_k \in E_T$. In the graph $G$, such a vertex $u$ is nonadjacent to at least one of the vertices $v_1, \ldots, v_{k-1}$. Say to vertex $v_j$ (otherwise, the clique $C$ could be augmented by vertex $u$ and would not be of maximum size as we assumed). In this case we may color $u$ with color $j$.

In the second case, the restriction of $T$ to $C$ does not form a star. In this case the restriction of $T$ to $C$ has a proper 2-coloring $C = C_1 \cup C_2$ with $|C_1| = a \geq |C_2| = b \geq 2$. Then there exist a vertex $x \in C_1$ and a vertex $y \in C_2$ for which $xy \notin E_T$. Let $v_1, \ldots, v_a = x$ be an enumeration of the vertices in $C_1$, and let $y = v_{a+1}, \ldots, v_{a+b}$ be an enumeration of the vertices in $C_2$. For $i = 1, \ldots, a$ we color vertex $v_i$ with color $i + 1$. For $i = 1, \ldots, b$ we color vertex $v_{a+i}$ with color $a + \lambda + i - 1$. This yields a λ-backbone coloring of $C$ with colors in $\{2, \ldots, k + \lambda - 1\}$. We color each vertex $u \in I$ with color

$$\begin{cases} k + \lambda & \text{if } uv \in E_T \text{ and } v \in C_1 \\ 1 & \text{if } uv \in E_T \text{ and } v \in C_2. \end{cases}$$

This yields a λ-backbone $(k + \lambda)$-coloring of $(G, T)$, since the colors of a vertex $v_i$ with $i \in \{1, \ldots, a\}$ and of any vertex $u \in I$ such that $uv_i \in E_T$ have distance at least $k + \lambda - (i + 1) \geq k + \lambda - (k - 2 + 1) > \lambda$, and since the colors of a vertex $v_i$ with $i \in \{a + 1, \ldots, b\}$ and of any vertex $u \in I$ such that $uv_i \in E_T$ have distance at least $a + \lambda + i - 1 - 1 \geq k/2 + \lambda - 1 \geq \lambda$.

**Proof of the tightness of the bounds.** The case $k = 1$ and $k = 2$ are trivial. For $k \geq 3$, we consider a split graph with a clique of $k$ vertices $v_1, \ldots, v_k$ and with an independent set of $(k - 2)(k - 1)/2$ vertices $u_{i,j}$ with $1 \leq i < j \leq k - 1$. Every vertex $u_{i,j}$ is adjacent to all vertices $v_s$ with $s \neq i$. The tree backbone $T$ contains the $k - 1$ edges $v_kv_s$ with $1 \leq s \leq k - 1$. The vertices $u_{i,j}$ form the leaves of $T$; in the tree, vertex $u_{i,j}$ is adjacent only to $v_j$. Clearly, $\chi(G) = k$.

Suppose to the contrary that $\BBC_\lambda(G, T) \leq k + \lambda - 1$, and consider such a backbone coloring. The vertices $v_1, \ldots, v_k$ in the clique must be colored with $k$ pairwise distinct colors. Since they form a star, either vertex $v_k$ has color 1, and colors 2, $\ldots, \lambda$ are not used on the clique, or vertex $v_k$ has color $k + \lambda - 1$, and colors $k, \ldots, k + \lambda - 2$ are not used on the clique. Both cases are symmetric, and we assume without loss of generality that $v_k$ has color
$k+\lambda -1$ and that colors $k, \ldots, k+\lambda -2$ are not used on the clique. Let $v_i$ be the vertex that has color $k-2$, and let $v_j$ be the vertex that has color $k-1$. The vertex $u_{i,j}$ is adjacent to all clique vertices except $v_i$; hence, it could only be colored with color $k-2$ or with a color in $\{k, \ldots, k+\lambda -2\}$. But these $\lambda$ colors are forbidden for $u_{i,j}$, since in the tree backbone it is adjacent to vertex $v_j$ with color $k-1$. Since there is no feasible color for $u_{i,j}$, we arrive at the desired contradiction.

3. Acknowledgments

This research was supported by the Research Fund of Institut Teknologi Bandung, Program: Riset Unggulan ITB 2006.

References