EFFECT OF COMBINATION OF DISPERSION AND NON LINEAR TERM TO DOWNSTREAM RUNNING NONLINEAR WATER WAVES

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Abstract.

This paper discuss the impact of dispersion and non-linear terms combinations the surface wave equation particularly on the peaking phenomena of the wave water that initially in the form of a bichromatic wave. The study for both of these terms are focused on the position where the bichromatic wave experience its highest peaking and its related bichromatic amplitude amplification. In the previous study, the position where the bichromatic wave experience its highest peaking is of order $O\left(\frac{1}{q^2},\frac{1}{\kappa^2}\right)$ and its bichromatic amplitude amplification is of order $O\left(\frac{q}{\kappa}\right)^4$, where q and κ_{lin} are the amplitude and frequency of the bichromatic wave envelope, respectively. This result is based on the fifth order Korteweg de Vries (KdV) equation and the quantity that obtained is called *Maximal Temporal Amplitude* (MTA). However, despite the the position where the bichromatic wave experience its highest peaking suits the result of Stansberg experiment and the result of numerical calculation using HUBRIS, its related bichromatic wave amplitude amplification is not close enough. The source of this discrepancy is suspected from the dispersion and non-linear terms of the KdV equation used. This study shows that the existence of the dispersion and non-linear terms influences the position of *Maximal Temporal Amplitude* (MTA) and its bichromatic wave amplitude amplification. For the coefficient of dispersion term of 1.0065 and nonlinear term of $\frac{3}{2}$, the position of MTA and bichromatic wave amplitude amplification suits the result of Stansberg experiment and the result of numerical calculation using HUBRIS.

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Introduction

This paper focuses on phenomena highest peaking of bi-chromatic waves during their propagation in hydrodynamics laboratory. The phenomena are very likely to occur because characteristic of bichromatic waves tend to unstable during their propagation, as mentioned by Zakharov [1]. The research that produces this paper was motivated by the needs of hydrodynamic laboratory aims to generate extreme waves in the wave tank at the desired position. Generated extreme waves will be used to test the floating objects, such as ships and other offshore construction, before the actual operating conditions. In [2,3], an extreme wave, which is also known as the giant wave, defined as the wave that has a height more than 2.2 times the average height of the waves around it. It is known that the extreme wave is unique waves. Besides it rare occurs, it also cannot be predicted. However, impact of these waves can cause significant damage to the objects around them, as described by Earle [4], Mori et al [5], Divinsky and Levin [6], Truslen and Dysthe [6], Smith [8], and Toffoli and Bitner [9]). Because it, researchers have done a lot of research on extreme wave, especially to understand the propagation and phenomena of appearance of the wave.

In general, properties of water waves are influenced by dispersion behavior and nonlinearity of water medium [10-13], during their propagation. Both of these properties lead to a shape changing of water wave as space and time function [12,13]. Splitting and peaking of waves during its propagation mark this shape change. Much of researches on waves, especially bichromatic waves, either by experimental, analytical or numerical have been done. Based on experimental results, Stansberg [14] and Westhuis [15,16] conducted a numerical study of the bichromatic wave. It is obtained that the highest splitting and peaking occur of bichromatic wave depends on two wave parameters, namely amplitude and frequency. Related to this, Marwan and Andonowati [13], by using a third-order approximation of KdV equation and a quantity called as MTA, founded that the position where bichromatic wave reaches highest peaking of order $O(\frac{1}{\ ^2},\frac{1}{\ ^2})$. It is seen that that position also depends on two wave parameters mentioned. Obtained results with that approximation in accordance with the results of Stansberg's experiment [14] and Westhuis' numerical study [15,16], as described in Marwan [17]. However, although it can predict the position, a third-order approximation of KdV equation did not to be able to predict an increase of, either bichromatic or Benjamin-Feir, wave amplitude (see Marwan [17,18] and Ramli [19]). In the other word, obtained results do not appropriate with the Stansberg's experimental results [14] and Westhuis' numerical study [16] to predict an increase of bichromatic wave amplitude. In the beginning, this incompatibility is expected because calculation is done to third-order only. Therefore, Ramli, et al [20] did a study about bichromatic wave propagation with fifth-order of KdV equation and MTA. Obtained results show that there is an increase of the related bichromatic wave amplitude as a high order influence. Nevertheless, the increasing is still not suitable with Stansberg's experimental result [14] and Westhuis' numerical study [16]. It should be stated that the KdV equation used in conducted study is KdV equation with exact dispersion relation found by Groesen [21].

This paper discuss an influence of the existence of the dispersion and the non linear terms toward an increase of bichromatic wave amplitude. Here bichromatic waves are superposition of two monochromatic waves that have same amplitude but slightly different of frequencies. Model used is KdV equation found by Cahyono [22] and a quantity called Maximal Temporal Amplitude (MTA), as mentioned in Marwan [13]. With this model is expected to able to be found dispersion and non linear terms of KdV equation, so that calculation results can be suitable with, either experimental or numerical results that have done before.

This paper was organized as the following. In section 2 will be presented mathematical model used and solution model up to third order. Deriving a period of maximum amplitude will be expressed in section 3. Some ilustrations for bi-chromatic case with various value of dispersion and nonlinear term will be described in this section. Finally this paper ends with conclusion.

Mathematical Model

Long wave model gratification with small amplitude that propagates in one direction at the surface can be represented by the KdV equation. In normalized variable, KdV equation can be written in the form [22]

$$\eta_t + \partial_x R\left[\eta + \frac{1}{2}R^{-1}(\eta.(R^{-1}\eta)) + \frac{1}{4}(R^{-1}\eta)^2\right] = 0. \qquad (1)$$

Here R is pseudo-differential operator with symbol

$$\hat{R}(k) = \sqrt{\frac{g\tanh k}{k}},$$

η denotes wave elevation, x and t are, spatial and time variables, respectively. In this paper, it was assummed that equation (1) can be expressed as

$$\partial_t\eta + (1+c_1)\Omega(-i\partial_x)\eta + \frac{c_0}{2}\partial_x\eta^2 = 0, \qquad (2)$$

called modified KdV (mKdV). In Groesen [21], values of c_0 and c_1 are $\frac{3}{2}$ and 0, respectively. Dispersion relation which is relate frequency and wave number is denoted by $\omega = (1+c_1)\Omega(k) = (1+c_1)k\sqrt{\frac{\tanh k}{k}}$. That relation is called as exact dispersion relation. Relation laboratory variables η_{lab}, x_{lab}, t_{lab} and normalized variables can be obtained through transformation $\eta_{lab} = h\eta$, $x_{lab} = hx$, and $t_{lab} = t\sqrt{h/g}$ with h is depth and g is gravity accelaration. Spatial variable x denotes horizontal direction and t is time variable.

As the research that has been conducted before [13,17,18], in this paper, an approximation for the solution equation (2) was determined by using asymtotic ekspantion in the power series form up to third-order toward elevation amplitude, which is written

$$\tilde{\eta} = \varepsilon^1\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta_{sb}^{(3)} \qquad (3)$$

with ε adalah a positive number, until first order represents amplitude of wave elevation. Form $\eta^{(1)}, \eta^{(2)}$, and $\eta_{sb}^{(3)}$, respectively, are linear solution, non-linear second order and non-linear third-order side band for wave elevation. It has been known that this method will produce resonance part in third-order. To avoid them, corecction is needed to conduct to wave number through development technique of Linstead-Poincare [23]

$$k^{nl} = k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)}. \qquad (4)$$

Because this paper focuses on bichromatic wave, first order solution was selected bi-chromatic wave that is written in the following form

$$\eta^{(1)} = q(e^{i\theta_+} + e^{i\theta_-}) + c.c \qquad (5)$$

with $\theta_+ = k_+ x - \omega_+ t,\ \omega_+ = (1 + c_1)\Omega(k_+^{(0)})$, where *c.c* denotes complex conjugate.

Furthermore, substitution (4) and (5) into (2) and equating coefficients ε that have the same power are obtained $\eta^{(1)}$ like in (4). At second order (ε^2) it is obtained $k_+^{(1)} = 0$ and

$$\eta^{(2)} = q^2\left(s_+ e^{2i\theta_+} + s_- e^{2i\theta_-} + s e^{i(\theta_+ + \theta_-)} + s_0 e^{i(\theta_+ - \theta_-)}\right) + c.c \qquad (6)$$

where

$$s_+ = c_0 \frac{k_\pm}{2\omega_\pm - (1 + c_1)\Omega(2k_\pm)}$$

$$s = c_0 \frac{k_+ + k_-}{\omega_+ + \omega_- - (1 + c_1)\Omega(k_+ + k_-)}$$

$$s_0 = c_0 \frac{k_+ - k_-}{\omega_+ - \omega_- - (1 + c_1)\Omega(k_+ - k_-)}$$

Resonance part at third-order produce a definition about non-linear dispertion relation $k_\pm^{nl} = k_\pm^{(0)} + k_\pm^{(2)}$

where

$$k_\pm^{(2)} = -\frac{q^2 c_0}{c_1} \frac{k_\pm^{(0)}}{V_g(k_\pm^{(0)})}(s_\pm + s + s_0) \qquad (7)$$

here $V_g(k_+^{(0)}) = (1 + c_1)\Omega'(k_+^{(0)})$ is group velocity. Besides that, the thrid-order non-linear resonance part has a form called third-order *side band* with frequency is almost same with first-order frequency. Third-order side band is expressed in the form,

$$\eta_{sb}^{(3)} = q^3\left(a_+ e^{i(2\theta_+ - \theta_-)} + a_- e^{i(2\theta_- - \theta_+)}\right) + c.c \qquad (8)$$

where

$$a_\pm = c_0^2(s_\pm + s_0)\frac{2k_\pm - k_\mp}{2\omega_\pm - \omega_\mp - (1 + c_1)\Omega(2k_\pm - k_\mp)}$$

This paper focuses on a solution that satisfies the initial conditions at wave maker, $x = 0$, $\tilde{\eta}(0,t) = q(e^{i\phi_+} + e^{i\phi_-}) + c.c$, as mentioned in [13,17,18]. Therefore, here the problem is boundary value problem

$$\partial_t \eta + (1 + c_1) i\Omega(-i\partial_x \eta) + \frac{c_0}{2}\partial_x \eta^2 = 0$$

With boundary conditions

$$\tilde{\eta}(0,t) = q(e^{i\phi_+} + e^{i\phi_-}) + c.c \qquad (9)$$

here $\phi_+ = -\omega_+ t$. To satifies the initial conditions at wave maker, the existence of the second and third-order part at that position tersebut has to be compensated with a part that called as *free waves*. *Free waves*, either second order or third-order satifiy linear dispersion relation. The secod order and third-order *side band* of freewaves are written

$$\eta_{fw}^{(2)} = c_0 q^2\left(s_+ e^{2i\theta(\omega_+)} + s_- e^{2i\theta(\omega_-)} + 2s e^{i\theta(\omega_+ + \omega_-)} + s_0 e^{i(\omega_+ - \omega_-)}\right) + c.c \qquad (10)$$

and

$$\eta_{sb\_fw}^{(3)} = q^3\left(a_+ e^{i\theta(2\omega_+ - \omega_-)} + a_- e^{i\theta(2\omega_- - \omega_+))}\right) + c.c \qquad (11)$$

with $\theta(\omega) = \Omega^{-1}\left(\frac{\omega}{\ldots}\right)x - \omega t$.

***Maximal Temporal Amplitude* and Its Maximum Position**

As mentioned in Groesen, et al [10], Westhuis, et al [15,16], Stansberg [14] and Zakharov [1], as effect of nonlinearity, the surface water waves experience the phenomena of peaking and splitting during their propagation in hydrodynamics laboratory. This phenomena, especially in peaking, can be observed using a quantity called the MTA as reported in [12,13]. The scale used to measure the height of the wave at each position, which is defined as

$m(x) = \max \eta(x,t), 0 < x < L,$

L specifies the length of the wave tank [20]. Then, the comparison of MTA at extreme position and the initial position called the amplitude amplification factor (AAF), which is defined as

$AAF = \frac{m(x_{max})}{\text{[illegible]}},$

with x_{max} represent the position where MTA reach maximum [20].

Using the same procedure as in [13,17,18], it can be shown that the modulated length of the carrier wave is

$\lambda_c = \frac{\text{[illegible]}}{\text{[illegible]}}$

with $\bar{k}^{nl} = (k_+^{nl} + k_-^{nl})/2$ and $\bar{K} = (\theta(\omega_+) + \theta(\omega_-))/2$. Consequently, the position where bishromatic waves experience the highest peaking can be written as

$$x_{max} = \frac{\pi}{|\bar{k}^{nl} - \bar{K}|} \qquad (12)$$

$$= \frac{\pi}{\left|2\nu^2\beta + 2q^2\left(\gamma + 2c_0^2\sigma_2\bar{k}/V_g(\bar{k})\right)\right|}$$

with $\beta = \frac{1}{2}K''(\bar{\omega})$ [21].

The following is presented some results of calculation and graphics by using the third order approximation of the mKdV equation as expressed in equation (2). Values assigned use MKS system (m,kg,s). Comparisons with some previous approximations that have been studied are presented also.

As an example case, it was used a bi-chromatic waves generated in the laboratory hydrodynamics with the depth and length of, respectively, 5 m and 250 m. Table 2 shows the calculation result of maximum position and amplitude amplification of bichromatic waves using mKdV equation for several values of c_0 and c_1. It shows that the combination of both c_0 and c_1 influence the maximum position and the amplitude amplification of bichromatic waves. It gives justification that the existences of dispersion term and non linear term influence the calculation of maximum position and the amplitude amplification of bichromatic waves. Furthermore, Figure 1 shows MTA graph calculated using third order mKdV equation in (2) for several values of c_0 and $c_1$based on Table 2. It shows that MTA having the value $c_0 = \frac{3}{2}$ and $c_1 = 0.0065$ (red) has the largest MTA compare to the MTA for the values of $c_0 = \frac{3}{2}$ and $c_1 = 0.005$ (black) as well as $c_1 = 0$ (blue). If consider from its maximum position, the MTA calculated using third order mKdV equation (2) using $c_0 = \frac{3}{2}$, $c_1 = 0.0065$ (red), $c_1 = 0.005$ (black), $c_1 = 0$ (blue) all show almost similar position, as shown in Table 2.

Table 1. Bichromatic wave parameters in laboratory scale and parameter in the normalized scale.

Parameter	Laboratory Scale	Normalized Scale
$4q$	0.16 m	0.0320
$\bar{\omega}$	3.145 rad/s	2.2464
ν	0.155 rad/s	0.1107
$\bar{k}_{lin}$	1.0185 rad/m	5.0923
κ_{lin}	0.1005 rad/m	0.5025

Table 2. The maximum position of the MTA and amplification factor for some values of c_0 and c_1 with $a = 0.04$, $\omega_+ = 3.3$, and $\omega_- = 2.99$

Case	$c_0 = \frac{3}{4}$	$c_0 = \frac{3}{2}$	$c_0 = 3$
$c_1 = -\frac{1}{2}$	$x_{max} = 1.5$ $AAF = 57.18$	$x_{max} = 2.5$ $AAF = 226.82$	$x_{max} = 3.5$ $AAF = 905.63$
$c_1 = -0.0067$	$x_{max} = 143.39$ $AAF = 1.03$	$x_{max} = 116.36$ $AAF = 1.089$	$x_{max} = 66.34$ $AAF = 1.34$
$c_1 = -0.0066$	$x_{max} = 143.41$ $AAF = 1.023$	$x_{max} = 116.39$ $AAF = 1.09$	$x_{max} = 66.36$ $AAF = 1.34$
$c_1 = -0.0065$	$x_{max} = 143.44$ $AAF = 1.02$	$x_{max} = 116.41$ $AAF = 1.09$	$x_{max} = 66.38$ $AAF = 1.34$
$c_1 = 0$	$x_{max} = 145.28$ $AAF = 1.04$	$x_{max} = 118.22$ $AAF = 1.16$	$x_{max} = 67.75$ $AAF = 1.61$
$c_1 = 0.0065$	$x_{max} = 147.10$ $AAF = 1.25$	$x_{max} = 120.02$ $AAF = 1.97$	$x_{max} = 69.12$ $AAF = 4.83$
$c_1 = 0.0066$	$x_{max} = 147.13$ $AAF = 1.27$	$x_{max} = 120.05$ $AAF = 2.07$	$x_{max} = 69.13$ $AAF = 5.22$
$c_1 = 0.0067$	$x_{max} = 147.16$ $AAF = 1.3003$	$x_{max} = 120.07$ $AAF = 2.20$	$x_{max} = 69.16$ $AAF = 5.72$
$c_1 = 1$	$x_{max} = 638.56$ $AAF = 1.0$	$x_{max} = 642.89$ $AAF = 1.0002$	$x_{max} = 660.79$ $AAF = 1.0007$

Figure 1. Plots of Maximal Temporal Amplitude for several values of c_1 with $c_0 = 3/2$, $\omega_+ = 3.3$, $q = 0.04$, and $\omega_- = 2.99$. c_1=0.0065 (red), c_1=0.005 (black) and c_1=0 (blue)

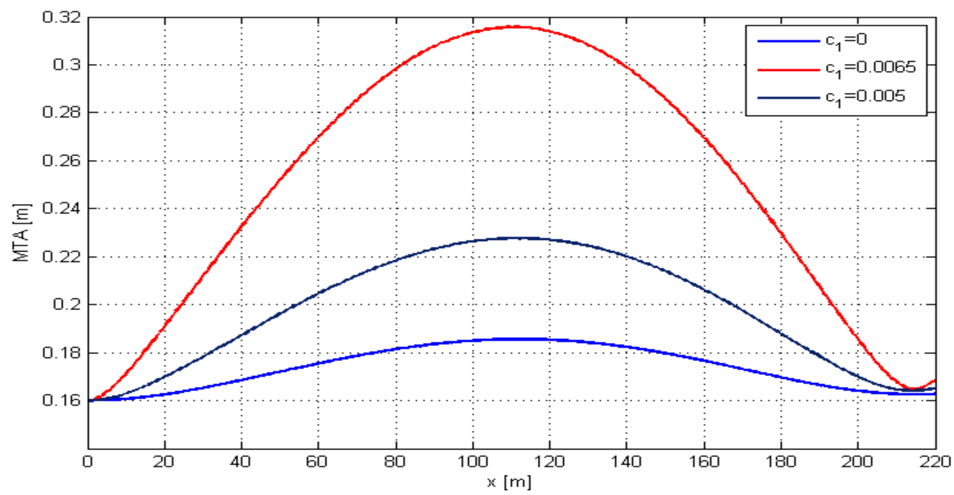

Figure 2. Plots of Maximal Temporal Amplitude for $c_0 = 3/2$, $\omega_+ = 3.3$, $q = 0.04$, and $\omega_- = 2.99$ were calculated by using some approximations. HUBRIS (red), third-order mKdV with $c_1 = 0.0065$ (purple), fifth-order KdV (green) and third-order KdV (blue).

Figure 2 shows MTA graph for $c_0 = 3/2$, $\omega_+ = 3.3$, $q = 0.04$, and $\omega_- = 2.99$ calculated using four methods : HUBRIS (red), third order mKdV with $c_1 = 0.0065$ (violet), fifth order KdV (green) and third order KdV (blue). It shows that the maximum position of MTA calculated using all four methods produces almost the same results. However, the maximum value of MTA calculated usning all four methods produces result that significantly different. Based on the result shows in Table 2, the maximum value of MTA for $c_0 = 3/2$, $\omega_+ = 3.3$, $q = 0.04$, and $\omega_- = 2.99$ calculated using thir order mKdV with $c_1 = 0.0065$ suits the result of numerical calculation in HUBRIS as well as the result of Stansberg experiment. This indicate the needs for further study regarding the dispersion term and non linear term of KdV equation that was found by Groesen [21] .

Concluding Remarks

This study has investigated the influence of dipsersion term and non linear term on the surface waves equation over the peaking phenomena of bichromatic waves. The study is conducted using KdV equation having pseudo differential operator and a quantiry called MTA. It found that there is influence of dispersion term and non linear term over the maximum position and the amplitude amplification of the bichromatic waves. For the dispersion term coefficient of 1.0065 and tne nonlinear term coefficient of $\frac{3}{7}$ of the mKdV used, it found that the maximum position and amplitude amplification of bichromatic waves calculated using third order mKdV and its MTA suits the result of of numerical calculation in HUBRIS as well as the result of Stansberg experiment. Therefor, it needs further study to investigate the dispersion term and non linear term of KdV eqaution that was found in the previous studies.

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